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On the system of non-linear differential equations $\dot{y}_k = y_k(y_{k+1} - y_{k-1})$

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Abstract. It is shown that the system of non-linear differential equations $\dot{y}_k = y_k(y_{k+1} - y_{k-1})$ ($k = 1, 2, \ldots; y_0 = 0$) can be reduced to the system of linear differential equations $\dot{\delta}_k = 4\delta_{k+1}$ ($k = 1, 2, \ldots$).

Using an inverse scattering technique, Kac and van Moerbeke [1] have studied the system of non-linear differential equations

$$\dot{R}_k = \exp(-R_{k-1}) - \exp(-R_{k+1}) \qquad (k = 1, 2, \ldots; R_0 = \infty) \tag{1}$$

under the restriction that $R_k \to 0$ as $k \to \infty$.

This system can be regarded as a discretisation of a certain non-linear wave equation (the Korteweg-de Vries equation) on the half line [2]. In the doubly infinite ($k = 0, \pm 1, \pm 2, \ldots$) case, it describes the propagation of a spectral packet of Langmuir oscillations on the background of thermal noise in a plasma and can also be treated by the inverse scattering method [3]. Kac and van Moerbeke [1] showed that (1) is transformed to another system of non-linear differential equations

$$\dot{\mu}_{2k} = 4(\mu_{2k+2} - \mu_2\mu_{2k}) \qquad (k = 0, 1, 2, \ldots; \mu_0 = 1). \tag{2}$$

The purpose of this paper is to derive the same equations as (2) without the assumption of $R_k \to 0$ ($k \to \infty$), and also to notice that they can be reduced to the following system of linear differential equations:

$$\dot{\delta}_k = 4\delta_{k+1} \qquad (k = 1, 2, \ldots) \tag{3}$$

which can be regarded as a discretisation of the linear wave equation ($\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$) on the half line. By setting $y_k = \exp(-R_k)$, (1) can be rewritten as

$$\dot{y}_k = y_k(y_{k+1} - y_{k-1}) \qquad (k = 1, 2, \ldots; y_0 = 0). \tag{4}$$

We shall consider (4) with $y_k (\neq 0) \in \mathbb{R}$.

It is well known [4] that to each sequence $\{y_k\}_1^\infty$ ($y_k \neq 0$) one can associate another sequence $\{\nu_k\}_1^\infty$ ($\nu_0 = 1$) by considering the finite continued fractions

$$f_0(\lambda) = \cfrac{1}{\lambda - \cfrac{y_1}{\lambda - \cfrac{y_2}{\lambda - \ddots \cfrac{y_{N-1}}{\lambda}}}} \qquad (N = 2, 3, \ldots) \tag{5a}$$

and their expansions in powers of λ^{-1}

$$f_0(\lambda) = \sum_{0 \leq k \leq N-1} \nu_k/\lambda^{2k+1} + \sum_{N \leq k < \infty} \nu'_k/\lambda^{2k+1} \tag{5b}$$

where the coefficients ν'_k $(k \geqslant N)$ are functions of ν_k $(0 \leqslant k \leqslant N-1)$. The y_k are expressed in terms of ν_l $(0 \leqslant l \leqslant k)$ as

$$y_k = P_{k-2}P_{k+1}/P_{k-1}P_k \tag{6}$$

with

$$\begin{aligned} P_{2k-1} &= \det\{(p_{2k-1})_{l,m}\} \qquad (p_{2k-1})_{l,m} \equiv \nu_{l+m-2} \qquad (1 \leqslant l, m \leqslant k) \\ P_{2k} &= \det\{(p_{2k})_{l,m}\} \qquad (p_{2k})_{l,m} \equiv \nu_{l+m-1} \qquad (1 \leqslant l, m \leqslant k) \end{aligned} \tag{7}$$

$(P_0 = P_{-1} = 1,\ P_k \neq 0\ (k = 1, 2, \ldots))$, which is also an immediate consequence of known results [4].

We shall later show that the time derivative of $f_0(\lambda)$ satisfies

$$\dot{f}_0 = 4[(\lambda^2 - y_1)f_0 - \lambda + \lambda(\hat{x}_N - \hat{z}_N)f_0] \tag{8}$$

with

$$\hat{x}_N = \cfrac{y_N}{\lambda - \cfrac{y_{N-1}}{\ddots \, \lambda - \cfrac{y_2}{\lambda - \cfrac{y_1}{\lambda}}}} \tag{9}$$

$$\hat{z}_N = \cfrac{y_N}{\lambda - \cfrac{y_{N-1}}{\lambda - \ddots \, \lambda - \cfrac{y_2}{\lambda}}}. \tag{10}$$

If we insert (5b) into the LHS of (8), it becomes

$$\dot{f}_0 = \sum_{1 \leqslant k \leqslant N-1} \dot{\nu}_k/\lambda^{2k+1} + \mathrm{O}(1/\lambda^{2N+1}) \tag{11}$$

while on the RHS,

$$(\lambda^2 - y_1)f_0 - \lambda = \sum_{0 \leqslant k \leqslant N-2} (\nu_{k+1} - y_1\nu_k)/\lambda^{2k+1} + (\nu'_N - y_1\nu_{N-1})/\lambda^{2N-1} + \mathrm{O}(1/\lambda^{2N+1}).$$

It will also be shown that

$$\lambda(\hat{x}_N - \hat{z}_N)f_0 = y_1y_2 \ldots y_N/\lambda^{2N-1} + \mathrm{O}(1/\lambda^{2N+1}) \tag{12}$$

$$\nu'_N + y_1y_2 \ldots y_N = \nu_N. \tag{13}$$

The RHS of (8) is then reduced to

$$\sum_{0 \leqslant k \leqslant N-1} 4(\nu_{k+1} - y_1\nu_k)/\lambda^{2k+1} + \mathrm{O}(1/\lambda^{2N+1}).$$

Comparing this with (11), we recognise that

$$\dot{\nu}_k = (4\nu_{k+1} - \nu_1\nu_k) \qquad (k = 1, \ldots, N-1;\ \nu_1 = y_1). \tag{14}$$

Since N is arbitrary, (14) is equivalent to (2).

To see that (2) can be reduced to (3), one has only to set

$$\mu_{2k}(= \nu_k) = \delta_{k+1}/\delta_1 \qquad (k = 0, 1, 2, \ldots).$$

To derive (8), we define k by k matrices $L_{N,k}$ and $B_{N,k}$ $(1 \leqslant k \leqslant N)$ by

$$(L_{N,k})_{l,l+1} = (L_{N,k})_{l+1,l} = \alpha_{N-l} \qquad (1 \leqslant l \leqslant k-1)$$
$$(L_{N,k})_{m,n} = 0 \qquad \text{(otherwise)}$$
$$(B_{N,k})_{l,l+2} = -(B_{N,k})_{l+2,l} = -2\alpha_{N-l}\alpha_{N-l-1} \qquad (1 \leqslant l \leqslant k-2)$$
$$(B_{N,k})_{m,n} = 0 \qquad \text{(otherwise)}$$

where

$$\alpha_k(\neq 0;\ k = 1, 2, \ldots) \in \mathbb{C} \qquad (y_k \equiv 4\alpha_k^2)$$

evolves according to

$$\dot{L}_{N,k} = B_{N,k}L_{N,k} - L_{N,k}B_{N,k} + H_{N,k} \tag{15}$$

with

$$(H_{N,k})_{1,2} = (H_{N,k})_{2,1} = 2\alpha_{N-1}\alpha_N^2$$
$$(H_{N,k})_{k-1,k} = (H_{N,k})_{k,k-1} = -2\alpha_{N-k+1}\alpha_{N-k}^2$$
$$(H_{N,k})_{m,n} = 0 \qquad \text{(otherwise)}.$$

We notice that if $L_{N,k}$ changes according to

$$\dot{L}_{N,k} = B_{N,k}L_{N,k} - L_{N,k}B_{N,k} \tag{16}$$

then

$$\Delta_{N,k} \equiv \det(E_{N,k} - L_{N,k}) \qquad (\Delta_{N,0} = 1,\ \Delta_{N,-1} = 0;\ N = 0, 1, 2, \ldots)$$

remains invariant ($\dot{\Delta}_{N,k} = 0$) [5]. When the time derivative of $L_{N,k}$ is prescribed by (15), it follows that

$$\begin{aligned}\dot{\Delta}_{N,k} &= -4\alpha_{N-1}^2\alpha_N^2\Delta_{N-2,k-2} + 4\alpha_{N-k-1}^2\alpha_{N-k}^2\Delta_{N,k-2} \\ &\quad + \text{(time derivative of } \Delta_{N,k} \text{ corresponding to the time change of (16))} \\ &= 4(y_{N-k}y_{N-k+1}\Delta_{N,k-2} - y_{N-1}y_N\Delta_{N-2,k-2}).\end{aligned} \tag{17}$$

For fixed N, we define f_k $(k = 0, 1, \ldots, N-1;\ f_N = 0)$ by

$$f_k = \Delta_{N,N-k-1}/\Delta_{N,N-k}.$$

Since $\Delta_{N,k}$ satisfies

$$\Delta_{N,k+1} = \lambda\Delta_{N,k} - y_{N-k}\Delta_{N,k-1} \qquad (k = 1, \ldots, N)$$

we recognise that

$$y_{k+1}f_kf_{k+1} = \lambda f_k - 1. \tag{18}$$

Hence

$$f_k = 1/(\lambda - y_{k+1}f_{k+1})$$
$$= \cfrac{1}{\lambda - \cfrac{y_{k+1}}{\lambda - \cfrac{y_{k+2}}{\ddots\, \lambda - \cfrac{y_{N-1}}{\lambda}}}}.$$

We also notice that

$$\Delta_{N,k} = \lambda\Delta_{N-1,k-1} - y_{N-1}\Delta_{N-2,k-2}. \tag{19}$$

By virtue of (17)-(19), $\dot{f}_0$ can be calculated as follows:

$$\begin{aligned}\dot{f}_0 &= 4[y_1y_2f_0f_1f_2 + y_{N-1}y_N(\Delta_{N-2,N-2}/\Delta_{N,N} - \Delta_{N-2,N-3}/\Delta_{N,N-1})f_0] \\ &= 4[y_1f_0(\lambda f_1 - 1) + \lambda y_N(\Delta_{N-1,N-1}/\Delta_{N,N} - \Delta_{N-1,N-2}/\Delta_{N,N-1})f_0] \\ &= 4[\lambda(\lambda f_0 - 1) - y_1f_0 + \lambda(\hat{x}_N - \hat{z}_N)f_0]\end{aligned}$$

where

$$\hat{x}_N \equiv y_N\Delta_{N-1,N-1}/\Delta_{N,N}$$

and

$$\hat{z}_N \equiv y_N\Delta_{N-1,N-2}/\Delta_{N,N-1}$$

can also be expressed as (9) and (10), respectively.

Proof of (12) and (13). We prove (12) by induction. For $N=1$, since $f_0 = 1/\lambda$, $\hat{x}_1 = y_1/\lambda$ and $\hat{z}_1 = 0$, (12) is obvious. We assume (12) for $N-1$. Then

$$\begin{aligned}\hat{x}_N &= y_N/(\lambda - \hat{x}_{N-1}) \\ &= (y_N/\lambda)[1 + (\hat{x}_{N-1}/\lambda) + \hat{x}_{N-1}^2 \mathrm{O}(1/\lambda^2)] \\ \hat{z}_N &= (y_N/\lambda)[1 + (\hat{z}_{N-1}/\lambda) + \hat{z}_{N-1}^2 \mathrm{O}(1/\lambda^2)]\end{aligned}$$

and

$$\begin{aligned}\lambda(\hat{x}_N - \hat{z}_N)f_0 &= \lambda(\hat{x}_{N-1} - \hat{z}_{N-1})f_0[(y_N/\lambda^2) + (y_N/\lambda)(\hat{x}_{N-1} + \hat{z}_{N-1})\mathrm{O}(1/\lambda^2)] \\ &= y_1y_2\ldots y_N/\lambda^{2N-1} + \mathrm{O}(1/\lambda^{2N+1}).\end{aligned}$$

By essentially the same calculation, we also find that

$$\begin{aligned}\nu_N - \nu'_N &= \lim_{\lambda\to\infty} \lambda^{2N+1}[f_0(\lambda; y_1, y_2, \ldots, y_N) - f_0(\lambda; y_1, y_2, \ldots, y_{N-1})] \\ &= y_1y_2\ldots y_N.\end{aligned}$$

We finally give some remarks on the relation between the method presented in this paper and the inverse scattering one [1]. Firstly, in the inverse scattering treatment the assumption that $R_k \to 0$ $(k\to\infty)$ is inevitable, as the name 'inverse scattering' suggests. In our method, on the other hand, this restriction is remedied. It is valid even for the complex variables $(y_k \in \mathbb{C})$ as long as $y_k \neq 0$ $(k = 1, 2, \ldots)$. Essentially, this method consists in the transformations of variables from $\{y_k\}_1^\infty$ to $\{\nu_k\}_1^\infty$ $(\nu_0 = 1)$ and from $\{\nu_k\}_1^\infty$ to $\{\delta_k\}_1^\infty$. The former is most concisely performed with the aid of finite continued fractions. In the special case of $y_k > 0$ $(k = 1, 2, \ldots)$, there exists a certain distribution function (spectral function) $\rho(\lambda)$ with infinitely many points of increase whose $2k$th moment coincides with ν_k:

$$\int_{-\infty}^{\infty} \lambda^{2k}\, \mathrm{d}\rho(\lambda) = \nu_k \qquad \int_{-\infty}^{\infty} \lambda^{2k+1}\, \mathrm{d}\rho(\lambda) = 0 \qquad (k = 0, 1, 2, \ldots).$$

In addition to this, if the assumption that $y_k \to 1$ $(k\to\infty)$, i.e. $R_k \to 0$ $(k\to\infty)$ is further imposed, the inverse scattering technique enables one to construct this distribution function from the scattering data whose time dependence can be integrated. For all these aspects, the reader is referred to [1, 4].

Secondly, even when the time dependence of distribution function $\rho(\lambda)$ is integrated by the inverse scattering method, the transformed system (2) is still non-linear. A further transformation leading to the linearised system (3) is therefore of great importance.

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